The Chaos of Damped-driven Pendulum System

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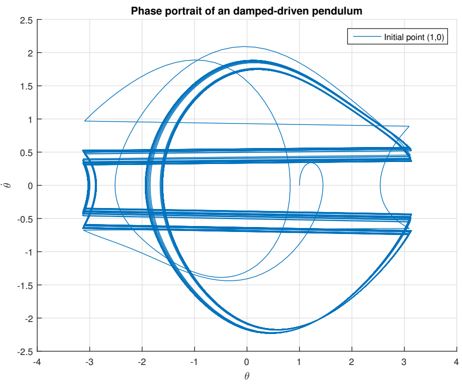

Figure 1: The left picture is the system with $q = 0.5, \omega = \frac{2}{3}, f = 0$. The right picture is the system with $q = 0.5, \omega = \frac{2}{3}, f = 1$. The initial points of both of them are (0,1).

Abstract

We will study some aspects of the damped driven pendulum. We will see that the trajectory of damped driven pendulum is unpredictable under certain ranges of parameters. Under some conditions, the trajectory behaves like random. A perturbation of initial condition will lead to different result. Several mathematical tools will be introduced in this study.

1 Introduction

In the 20th century, Edward Norton Lorenz created a mathematical model of weather. He found that when the initial condition is perturbed a little bit, the outcome is completely different. Even the differential equation system that looks simple,says pendulum. Under certain conditions, the

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outcome is unpredictable or more accurately, the system will become chaotic. Chaos has no rigorous definition in mathematics. If a system is said to be chaotic, it should satisft three property. Sensitive to initial conditions, topologically mixing and have dense periodic orbits. In this paper, we will discuss the behaviour of the damped driven pendulum. We will introduce Poincare section and bifurcations diagram. By plotting both of them, we will have a intuition on what chaos is.

2 Governing equations

The governing equation of damped driven pendulum equation is

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin\theta = f\cos\omega t\,. \tag{1}$$

This may be written as a system of coupled first-order equations:

$$\begin{aligned} \dot{\theta} &= u\,, \\ \dot{u} &= -\frac{1}{q}u - \sin\theta + f\cos\phi\,, \\ \dot{\phi} &= \omega \end{aligned} \tag{2}$$

3 Numerical Methods

To solve the system, we will use the Runge-Kutta 4th order methods. Define $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4$. The local truncation error is $O(h^5)$ and the global truncation error is $O(h^4)$, where h is the step size.

4 Require knowledge

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5 Results

5.1 Phase space diagram

A phase space is a space that in which all possible states of a dynamical system are represented.
the phase diagram of a damped driven pendulum with $q = 1, f = 1, \omega = 1$.

5.2 Poincare section

As we are modeling some behaviors in the real world, the solutions of differential equations should be consistent to the real world. The solution is a function of time. When time tends to infinity, the solution should be tend to a fixed point or finite number of points or some orbits. We call the solution set as an attractor. An attractor is strange if the solution set has a fractal property.
Poincare section is a useful tool that help us to distinguish chaos. We sample the point $(\theta, \dot{\theta})$ every regular time period. In our case, period $= \frac{2\pi}{\omega}$. The parameters are $f = 1.5, \omega = \frac{1}{3}, q = 4$, we computed 10000 points on the graph.

If we continue plotting, the picture is "denser". We assume it is a strange attractor. By numerical computing, we can estimate the degree of dimension of it.

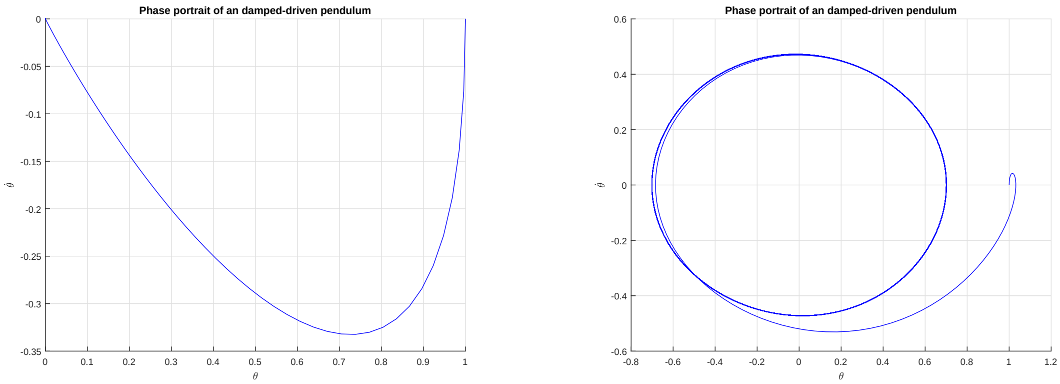

Figure 2: The left picture is the system with $q = 0.5, \omega = \frac{2}{3}, f = 0$. The right picture is the system with $q = 0.5, \omega = \frac{2}{3}, f = 1$. The initial points of both of them are (0,1).

5.3 Bifurcations diagram

Now we set q as the control parameter, with $f = 1.5, \omega = \frac{2}{3}$. When$q < 1.3485$, the system is stable 1 period cycle.
At $q = 1.3485$, the system becomes unstable and gives a succession of a pitchfork bifurcation.
When $1.3485 < q < 1.37$, the system undergoes 2 period cycle.
At $q = 1.37$, the system becomes unstable and gives a succession of a pitchfork bifurcation again.
At $q > 1.378$, the system becomes chaotic. It represents that the unpredictability. With small perturbation of initial condition, the result will be completely different.

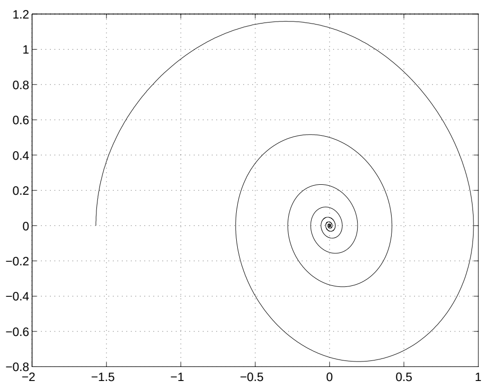

Figure 3: Example of the phase-space evolution of an underdamped pendulum. Note that the axises are not labeled and many space is wasted on both sides of the figure.

It is also possible to put two or more figures side-by-side. Suppose we wanted to display

Figures 4 (a) and (b) to demonstrate the difference in the phase portraits of a critically damped and overdamped pendulum ($q = 1/2$ and 1, respectively). You may look at the LaTeX to see how this is done.

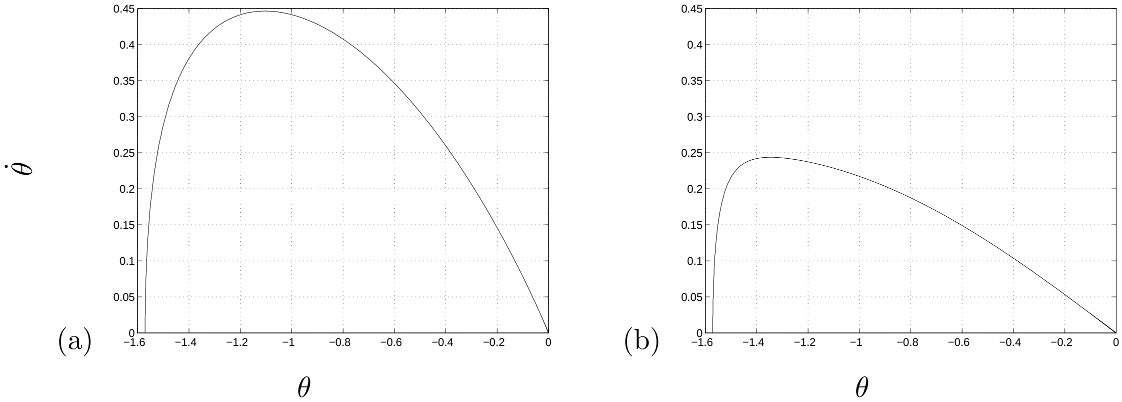

Figure 4: When placing figures side-by-side like this, be sure that text in the figure is large enough to be legible. Observed also that it is not straight-forward to add labels to the figure at this stage. You are advised to add them in MATLAB instead. Also, it might be a better presentation if one plots two curves on the same figure and have it placed next to Figure 3.

6 Conclusions

Here you discuss what you have learned from this work, and what the reader should have learned from your paper.

Acknowledgments

If there are some colleagues in or outside the class that you have discussed this work with, you may want to acknowledge these discussions here. Alternatively, if one of the TA's helped you extensively with your paper, this may be a good place to thank them.

References

[1] G. L. Baker, & J. P. Gollub, *Chaotic Dynamics*, Cambridge University Press, 1990.

[2] L. Lamport, *LaTeX: A document preparation system*, Addison Wesley Publishing Company, 1994.