Solutions to the Damped Oscillator Equation

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The Damped Oscillator Equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

is a second-order differential equation that can easily be derived using Newton's Second Law of Motions applied to a spring, assuming a frictional force $f_k = -\beta v$, where v is the speed of the mass. Solving this equation, however, is considerably more difficult, but not that hard if proper techniques are used.

1 Over-damped Oscillators

An over-damped oscillator is an oscillator that faces a frictional force so great that the oscillator just returns to its equilibrium position. Looking at the equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

It can be proved that if y_1 and y_2 are two linearly independent solutions, meaning that one cannot be multiplied by a number to get the other, then $y = c_1y_1 + c_2y_2$ must be a solution. Looking at properties of functions, e^{rx} seems like a good solution to the equation. If e^{rx} is indeed a solution, then plugging back into the main equation gives

$$mr^2e^{rx} + \beta re^{rx} + ke^{rx} = (mr^2 + \beta r + k)e^{rx} = 0$$

That means that if e^{rx} is a solution $mr^2 + \beta r + k = 0$. This is the auxiliary equation, and notice that we can get two roots, one root, or zero real roots. An over-damped oscillator will result in two roots in the equation. If r_1 and r_2 are solutions, then from the theorem above, the solution must be

$$x(t) = c_1e^{r_1x} + c_2e^{r_2x}$$

These two roots are

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

Thus,

$$x(t) = c_1 e^{\frac{-\beta+\sqrt{\beta^2-4mk}}{2m}t} + c_2 e^{\frac{-\beta-\sqrt{\beta^2-4mk}}{2m}t}$$

The constants c_1 and c_2 can be determined by the amplitude and the initial velocity of the oscillation.

2 Critically Damped Oscillators

A critically damped oscillator just slightly overshoots the equilibrium position and gently returns to equilibrium. In this case, the auxiliary equation derived in section 1

$$mr^2 + \beta r + k = 0$$

must have one solution. Let r be the one solution. Then

$$r = -\frac{\beta}{2m}.$$

However, we cannot use the solution above by putting r for both r_1 and r_2 as that would result in two linearly dependent terms. However, if we know that e^{rx} is a solution, then it can be verified that $y = xe^{rx}$ is a solution using substitution into the original equation and the fact that $2mr + \beta = 0$. Thus, a solution is

$$x(t) = c_1 e^{-\frac{\beta}{2m}t} + c_2 t e^{-\frac{\beta}{2m}t} = (c_1 + c_2 t)e^{-\frac{\beta}{2m}t}$$

3 Under-damped Oscillators

An under-damped oscillator will oscillate, but the amplitude will gradually decrease and the oscillations will gradually die out. The auxiliary equation

$$mr^2 + \beta r + k = 0$$

has no real solutions but two *complex* solutions. Let r_1 and r_2 be solutions. Let $r_1 = a + bi$ and $r_2 = a - bi$. Using the general solution from section 1, we can show that

$$x(t) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} = e^{at}(C_1 e^{ibt} + C_2 e^{-ibt})$$

Using Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can show that

$$x(t) = e^{at}(C_1(\cos bt + i\sin bt) + C_2(\cos bt - i\sin bt)$$

Simplifying, we get

$$x(t) = e^{at}[(C_1 + C_2)\cos bt + i(C_1 - C_2)\sin bt] = e^{at}(c_1\cos bt + c_2\sin bt)$$

where $c_1 = C_1 + C_2$ and $c_2 = i(C_1 - C_2)$. If we are looking for real solutions, then c_1 and c_2 must be real. Back to the original equation, we get

$$r = -\frac{\beta}{2m} \pm \frac{i\sqrt{4mk - \beta^2}}{2m}$$

Thus, the solution is

$$x(t) = e^{-\frac{\beta}{2m}t}\left[c_1 \cos\left(\frac{\sqrt{4mk-\beta^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk-\beta^2}}{2m}t\right)\right]$$