Integration of Some Elementary Integrals

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0.0.1 1

$$\int x\cos(ax)\ dx$$

Let $u = x \therefore du = dx$ and $dv = \cos(ax)dx \therefore v = \frac{1}{a}\sin(ax)$

$$= \frac{x\sin(ax)}{a} - \int \frac{1}{a}\sin(ax)\ dx$$

$$= \frac{x\sin(ax)}{a} + \frac{1}{a^2}\cos(ax) + C$$

0.0.2 2

$$\int \sin^2(x)\cos^2(x)\ dx$$

$$= \int (1+\cos(2x))(1-\cos(2x))\ dx$$

$$= \int (1-\cos^2(2x))\ dx$$

$$= \frac{1}{2}\int (2 - 2\cos^2(2x)\ dx$$

$$= \frac{1}{2}\int (\cos(4x) + 4)\ dx$$

$$= \frac{1}{32}(\sin(4x) + 4x)\ dx + C$$

0.0.3 3

$$\int \frac{7x+1}{x^2-x-2}\,dx$$

$$\frac{7x+1}{x^2-x-2} = \frac{7x+1}{(x-2)(x+1)}$$

$$7x+1 = \frac{A}{x-2} + \frac{B}{x+1}$$

$$7x+1 = Ax + A + Bx - 2B$$

$$7x+1 = (A+B)x + (A-2B)$$

$$\therefore A+B = 7.....(\text{i})$$

$$A-2B = 1....(\text{ii})$$

subtracting equation (ii) from (i) gives $\Rightarrow 3B = 6 \therefore B = 2$ and $A = 5$

$$\int \frac{7x+1}{x^2-x-2}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{2}{x+1}\,dx$$
$$= 5\ln|x-2| + 2\ln|x+1| + C$$

0.0.4 4

$$\int \frac{1}{(x+1)^2}\,dx$$

Let $u = x+1 \therefore du = dx$

$$= \int \frac{1}{u^2}\,du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{(x+1)} + C$$

0.0.5 5

$$\begin{aligned}\int \sin^6(x)\cos^3(x)\ dx &= \int \sin^6(x)\cos^2(x)\cos(x)\ dx \\ &= \int \sin^6(x)(1-\sin^2(x))\cos(x)\ dx \\ &= \int \sin^6(x)\cos(x)\ dx - \int \sin^8\cos(x)\ dx\end{aligned}$$

Let $u = \sin(x) \therefore du = \cos(x)$

$$\begin{aligned}&\int u^6\ du - \int u^8\ du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + C \\ &= \frac{\sin^7(x)}{7} - \frac{\sin^9(x)}{9} + C\end{aligned}$$

0.0.6 6

$$\int \sin(\theta)\ln(\cos(\theta))\ d\theta$$

Let $u = \ln(\cos(\theta)) \therefore du = -\tan(\theta)\ d\theta$

$dv = \sin(\theta)\ d\theta \therefore v = -\cos(\theta)$

$$\begin{aligned}\int \sin(\theta)\ln(\cos(\theta)\ d\theta &= -\cos(\theta)\ln(\cos(\theta)) - \int \sin(\theta) d\theta \\ &= -\cos(\theta)\ln(\cos(\theta)) + \cos(\theta) + C\end{aligned}$$

0.0.7 7

$$\int \sec^4(x)\tan^3(x)\ dx$$
$$= \int \sec^3(x)\tan^2(x)\sec(x)\tan(x)\ dx$$
$$= \int \sec^3(x)(\sec^2(x)-1)\sec(x)\tan(x)\ dx$$
$$= \int \sec^5(x)\sec(x)\tan(x)\ dx - \int \sec^3(x)\sec(x)\tan(x)\ dx$$
Let $u = \sec(x) \therefore du = \sec(x)\tan(x)\ dx$
$$= \int u^5\ du - \int u^3\ du$$
$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$
$$= \frac{\sec^6}{6} - \frac{\sec^4}{4} + C$$

0.0.8 8

$$\int xe^{5x}\ dx$$
Let $u = x \therefore du = dx$

and $dv = e^{5x} \therefore v = \frac{e^{5x}}{5}$
$$\frac{xe^{5x}}{5} - \int \frac{e^{5x}}{5}\ dx$$
$$= \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + C$$

0.0.9 9

$$\int \sec^3(\theta)\ d\theta$$

$$= \int \sec(x) \sec^2(x)\ dx$$

Let $u = \sec(x) \therefore du = \sec(x)\tan(x)\ dx$ and $dv = \sec^2(x)\ dx \therefore v = \tan(x)$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx$$

$$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx$$

$$= \sec(x)\tan(x) - \int \sec^3(x)\ dx + \int \sec(x)\ dx$$

$$= \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x) + \tan(x)|}{2} + C$$

0.0.10 10

$$\int \tan^3(x)\sec^4(x)\ dx$$
$$= \int \sec(x)\tan(x)\sec^3(x)\tan^2(x)\ dx$$
$$= \int \sec^5(x)\sec(x)\tan(x)\ dx - \int \sec^3(x)\sec(x)\tan(x)\ dx$$
Let $u = \sec(x)\ du = \sec(x)\tan(x)\ dx$
$$= \int u^5\ du - \int u^3\ du$$
$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$
$$= \frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + C$$

0.0.11 11

Evaluate where E is the region under the plane that lies in the first octant.

$$\int_0^3 \int_0^{\frac{-2x}{3}+2} \int_0^{6-2x-3y} 2x \; dz \; dy \; dx$$

$$= \int_0^3 \int_0^{\frac{-2x}{3}+2} 2xz \Big]_0^{6-2x-3y} \; dy \; dx$$

$$= \int_0^3 \int_0^{\frac{-2x}{3}+2} 2x(6-2x-3y) \; dy \; dx$$

$$= \int_0^3 \int_0^{\frac{-2x}{3}+2} (12x - 4x^2 - 6xy) \; dy \; dx$$

$$= \int_0^3 (12xy - 4x^2y - 3xy^2) \Big]_0^{\frac{-2x}{3}+2} \; dx$$

$$= \int_0^3 \left(12x\frac{6-2x}{3} - 4x^2\frac{6-2x}{3} - 3x\frac{36-24x+4x^2}{9} \right) \; dx$$

$$= \int_0^3 \left(\frac{36x(6-2x) - 12x^2(6-2x) - 108x + 72x^2 - 12x^3}{9} \right) \; dx$$

$$=$$

$$= \int_0^3 \left(\frac{216x - 72x^2 - 72x^2 + 24x^3 - 108x + 72x^2 - 12x^3}{9} \right) \; dx$$

$$= \int_0^3 \frac{-72x^2 + 108x + 12x^3}{9} \; dx$$

$$= \int_0^3 -8x^2 + 12x + \frac{4x^3}{3}$$

$$= \left[\frac{-8x^3}{3} + 6x^2 + \frac{x^4}{3} \right]_0^3$$

$$= 9$$

0.0.12 12

Prove that $\dfrac{d(\sin^{-1}(x))}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

Proof:

$$\text{Let } f(x) = \sin(x) \text{ and } g(x) = \sin^{-1}(x)$$

Then $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1}(x))}$
Let's $y = \sin^{-1}(x) \therefore x = \sin(y)$
Using this part of the definition:
$\cos(\sin^{-1}(x)) = \cos(y)$
But we know that:
$\sin^2(y) + \cos^2(y) = 1 \therefore \cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$
$\therefore \frac{d(\sin^{-1}(x))}{dx} = \frac{1}{\sqrt{1-x^2}}$

0.0.13 13

Prove that $\frac{d(\cos^{-1}(x))}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Proof:

$$\text{Let } f(x) = \cos(x) \text{ and } g(x) = \cos^{-1}(x)$$

Then $g'(x) = \frac{1}{f'(g(x))} = \frac{-1}{\sin(\cos^{-1}(x))}$
Let's $y = \cos^{-1}(x) \therefore x = \cos(y)$
Using this part of the definition:
$\sin(\cos^{-1}(x)) = \sin(y)$
But we know that:
$\sin^2(y) + \cos^2(y) = 1 \therefore \sin(y) = \sqrt{1-\cos^2(y)} = \sqrt{1-x^2}$
$\therefore \frac{d(\cos^{-1}(x))}{dx} = \frac{-1}{\sqrt{1-x^2}}$