Extra Credit Problem

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1 Problem

$let\, y = (Ax^2 + Bx + C)e^{Dx}$

$find\, A, B, C, D \in \mathbb{Z}\, such\, that\, y, \frac{dy}{dx}, \frac{dy^2}{dx^2}\, factor\, over\, \mathbb{Q}$

2 Solution

Proof.

$\frac{dy}{dx} = (ADx^2 + (BD + 2A)x + (CD + B))e^{Dx}$

$\frac{dy^2}{dx^2} = (AD^2x^2 + (BD^2 + 4AD)x + (CD^2 + 2A))e^{DX}$

$in\, order\, for\, these\, to\, factor,\, the\, quadratic\, elements\, must\, have\, roots\, in\, \mathbb{Q}$

$\therefore\, their\, respective\, discriminants\, must\, be\, perfect\, squares$

$\Delta\, of\, y = B^2 - 4AC = n_0^2\, where\, n_0 \in \mathbb{Z}$

$\Delta\, of\, \frac{dy}{dx} = B^2D^2 - 4ACD^2 + 4A^2 = (Dn_0)^2 + 4A^2 = n_1^2\, where\, n_1 \in \mathbb{Z}$

$\Delta\, of\, \frac{dy^2}{dx^2} = B^2D^4 - 4ACD^4 + 8A^2D^2 = n_2^2\, where\, n_2 \in \mathbb{Z}$

$(2A)^2 + (Dn_0)^2 = n_1^2$

$D^2((Dn_0)^2 + 8A^2) = n_2^2$

$(2A)^2 + n_1^2 = u^2\, where\, u = n_2/D,\, which\, must\, necessarily\, be\, in\, \mathbb{Z}$

$no\, solution\, for\, A, B, C, D \neq 0\, by\, Fermat's\, Right\, Triangle\, Theorem$

$\square$